Mutual Resistance in Spicelink

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1. Introduction

In this note, we discuss the “mutual resistance” phenomenon and investigate why it occurs. In order to do this, we briefly review some key concepts from electromagnetics and circuit theory. Armed with these results, we can show how the mutual resistance arises through the “proximity effect.” We can also show how mutual resistance can be modeled using circuit elements. Finally we discuss why mutual resistance is important for calculating the total resistance in real circuits.

2. Background

The physics of electromagnetism are governed by Maxwell’s equations for the electric and magnetic fields, $\vec{E}$ and $\vec{H}$, respectively. In the general high-frequency case we have

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \tag{1}$$

$$\nabla \times \vec{H} = \vec{J} + j\omega\varepsilon\vec{E} \tag{2}$$

Here, ε is the permittivity of the medium, μ is its permeability, and $\vec{J}$ is a current density due to the movement of free charges. The angular frequency is ω. Maxwell’s equations constitute a complete description of the electromagnetic fields from very low frequency to very high. Within certain types of materials, or in certain frequency ranges, Maxwell’s equations can be reduced to specialized forms that are more convenient to work with. One such form is the eddy current equation for good conductors.

2.1. The Eddy Current Equation and Associated Phenomena

Within a linear, isotropic conducting material, the current density is given simply by Ohm’s law

$$\vec{J} = \sigma\vec{E}$$

where σ is the conductivity of the material. For highly conductive materials such as copper, the value of σ is on the order of $5\times10^{7}\,\mathrm{S/m}$. This causes the conduction term (first term on the right hand side) of Maxwell’s second equation (2) to dominate over the displacement current term (second term on the right hand side) up to extremely high frequencies, on the order of $10^{18}\,\mathrm{Hz}$. Therefore, for electronic circuit designs it is generally safe to ignore the second term in (2) and rewrite it simply as

$$\nabla \times \vec{H} = \sigma \vec{E} \tag{3}$$

By taking the curl of (1) we can eliminate the magnetic field from (3), which gives the basic electromagnetic equation for good conductors:

$$\nabla \times \nabla \times \vec{E} + j\omega\mu\sigma\vec{E} = 0 \tag{4}$$

This is called the *eddy current equation*, because its solutions always consist of currents (and associated electric fields) that form closed loops, similar to the whirlpools that form when water is forced to flow around an obstacle. If there are external current sources applied to the conductor, then these loops may only close if we consider the external source to be part of the loop.

The solutions of the eddy current equation have some other well-known properties:

- *Skin effect*: At moderate to high frequencies, we observe that the current in the conductors flows almost entirely near its surface in a region whose thickness is called the *skin depth*. The skin depth δ decreases with increasing frequency according to the relationship

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \tag{5}$$

- *Surface impedance*: It can be shown that the total current J_s flowing beneath the surface of the conductor within the skin depth region can be related to the electric field E_s on the conductor surface by the relation $E_s = Z_s J_s$, where Z_s is the *surface impedance*, given by

$$Z_s = \sqrt{\frac{j\pi f \mu}{\sigma}} \tag{6}$$

  This is a microscopic phenomenon—relating the electric field at a point to the total current directly beneath that point. However, it gives rise to similar effects on a large scale: we observe that at a sufficiently high frequency, conductors exhibit a net resistance that increases as the square root of frequency.

- *Proximity effect*: The current flowing within a given conductor produces a magnetic field (see equation (2) above) that surrounds it. If another conductor is brought into the neighborhood of the first, these magnetic field lines impinge upon the second conductor. Equation (1) indicates that if the frequency is nonzero, these magnetic fields will induce an electric field (and hence a current) to develop within the second conductor. The currents that flow in the second conductor in response to the current in the first are called *induced currents*. If the frequency is moderate to high, then most of these induced currents flow near the surface in a skin depth region. In addition, these currents will tend to be strongest on the side of the second conductor closest to the first—a phenomenon known as the *proximity effect*.

2.2. Eddy Current Calculations in Spicelink

Both Maxwell Q3D Extractor (Versions 4.0 and 4.5) and Maxwell 2D Extractor (Versions 1.0 through 4.5) attempt to capture eddy current effects. Q3D does this when solving for AC Inductance/Resistance problems, and 2D Extractor performs eddy current calculations when solving for Impedance. The eddy current calculation in 2D Extractor is the more rigorous; a 2-dimensional version of the eddy current equation (4) is solved using a finite element method. The calculations in Q3D Extractor are not a true solution of (4), but are rather based upon the surface impedance formula in (6). That is, Q3D Extractor's AC Inductance/Resistance calculation assumes that the frequency is sufficiently high to approximate the conductor current as a surface current distribution. This surface current distribution must satisfy Maxwell's equations (1) and (2), with the restriction that the displacement current term in (2) is negligible. After a suitable set of surface currents has been calculated, the surface impedance formula is employed to calculate the resistance matrix for the structure. The process for resistance calculation is discussed in the following sections.

2.3. Calculation of AC Resistances for Single Conductor Problems

Recall from basic circuit theory that the power $p(t)$ dissipated in a circuit element at any instant in time t is given by the product $p(t) = v(t)i(t)$, where $v(t)$ and $i(t)$ are respectively the instantaneous voltage across the element and the current through it. For electromagnetic problems, power is dissipated throughout the body of the conductor, so we introduce the concept of a power density $p(x, y, z)$ at a given point in space. The power density is calculated in an analogous way to the power in circuit theory, by multiplying the instantaneous values of the electric field (analogous to voltage) and the current density (analogous to current.) This is known as *Joule's law*. The power density is measured in $\mathrm{W/m^3}$ rather than Watts.

The eddy current equation (4) is a complex phasor equation. The phasor $\vec{E}$ represents the direction, magnitude and phase of the electric field, which is assumed to be a time-varying, sinusoidal quantity. For a good conductor, the current $\vec{J} = \sigma\vec{E}$ is also a phasor, and is in phase with $\vec{E}$. To compute the time-average power being dissipated at a single point in space, we must multiply the instantaneous current density and electric field at that point, integrate over a single period, and then divide by the length of the period. The result of this calculation is the time average power density $P(x, y, z)$, given by

$$P(x, y, z) = \frac{1}{2}\left(\vec{J}^{*} \cdot \vec{E}\right) = \frac{1}{2}\sigma\left|\vec{E}\right|^2 \qquad (7)$$

Here, the (*) superscript denotes the complex conjugate operation. Equation (7) is similar to the formula from basic circuit theory for the average power dissipated in a circuit element:

$$P = \frac{1}{2}\mathrm{Re}(I^{*}V)$$

For a simple real-valued resistor, $V = RI$ and this equation reduces to $P = \frac{1}{2}R|I|^2$.

Having obtained the formula for the power dissipation at a single point, we can compute the total average power P being dissipated within a given conductor by integration. We have

$$P = \int_V \frac{1}{2}\sigma(x, y, z)\left|\vec{E}(x, y, z)\right|^2 dV. \tag{8}$$

where V denotes the volume of space occupied by the conductor.

Suppose that we have a problem that consists of a single conductor. We can attach a 1 Amp sinusoidal current source across the conductor and solve for the current distribution (or equivalently the electric field pattern) in it using a field solver. Then we can recover the total power dissipated within the conductor using (8). Now in order to compute the resistance of the conductor, the result from (8) must agree with the power dissipation expected from circuit theory, namely $\frac{1}{2}R|I|^2$. But since $|I| = 1$, this implies that

$$R = \int_V \sigma(x, y, z)\left|\vec{E}(x, y, z)\right|^2 dV. \tag{9}$$

Maxwell 2D Extractor uses exactly this sort of integral to compute the resistance of a single-conductor problem. Q3D uses a similar formula, but with surface currents and surface impedances replacing the electric field and conductivity:

$$R = \int_V \mathrm{Re}(Z_s)\left|\vec{J}_s(x, y, z)\right|^2 dV. \tag{10}$$

In the next section, we consider the problem of computing resistances (both self and mutual) in systems of multiple conductors.

3. Resistance Calculations for Multiple Conductor Problems

Consider now a system of two different conductors excited by two independent AC current sources I_1 and I_2 as shown below in Figure 1.

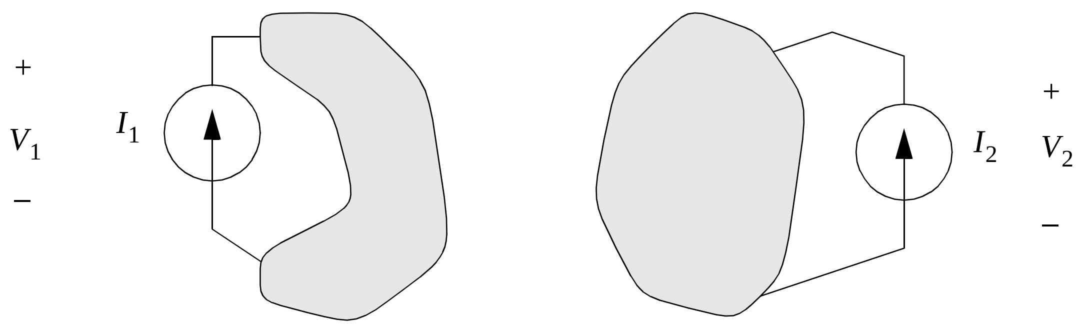

Figure 1. A pair of conductors excited by two independent AC current sources will be used to illustrate the mutual resistance concept.

If we consider for a moment the DC case, then we expect that the two conductors each have a self-resistance, which can be computed by turning on the associated current source and computing the power dissipation in that conductor. For example, we could set $I_1 = 1$ A and $I_2 = 0$ and compute the power dissipated in conductor 1. Then by applying (9), we can compute the self-resistance of conductor 1, R_{11}. Because we are at DC, there will be no induced currents in the other conductor, and hence no power dissipation there. Therefore it would be sufficient to carry out the integral in equation (9) over only the volume of conductor 1.

If we raise the frequency, then the story is somewhat different. Even if $I_2 = 0$, there will still exist some induced currents in conductor 2. This gives rise to some power dissipation in that conductor as well. This phenomenon is known as *induction heating*; it is often employed in industrial processes where precise amounts of heat are to be delivered to a conductive object.

Therefore we see that the true total power dissipation must be computed by extending the domain of the integral in (9) to *both* conductors. The added power loss in the second conductor will increase the self-resistance of conductor 1 above the value that it would have had in isolation. (Indeed, even if conductor 1 had infinite conductivity, it would still exhibit a non-zero self-resistance due to the presence of the nearby, finite-conductivity object!) Our new formula for power dissipation is

$$P = \int_{V_1 + V_2} \frac{1}{2}\sigma(x, y, z)\left|\vec{E}(x, y, z)\right|^2 dV \qquad (11)$$

where $V_1 + V_2$ denotes the combined volumes of conductors 1 and 2.

Consider now what occurs in the AC case when we set $I_1 = 1\text{A}$ and $I_2 = 0$. This produces distributions of electric field $\vec{E_1}$ and corresponding current density $\vec{J_1}$ that satisfy the eddy current equation (4). If instead we set $I_1 = 0$ and $I_2 = 1\text{A}$, we would get a different set of fields $\vec{E_2}$ and $\vec{J_2}$ that again satisfy (4) as well as the applied boundary conditions. Because we have assumed that the materials involved have linear conductivities (independent of the electric field), the field solutions are linear in I_1 and I_2. Therefore, if we apply an arbitrary combination of the two external current sources, we will get a total field solution that is a weighted sum of the two:

$$\vec{E} = \vec{E_1} I_1 + \vec{E_2} I_2$$

Let us now evaluate the power dissipation for this general problem. From (11) we have

$$P = \int_{V_1 + V_2} \frac{1}{2}\sigma(x, y, z)\left|\vec{E_1}(x, y, z)I_1 + \vec{E_2}(x, y, z)I_2\right|^2 dV$$

The squared magnitude of the sum of two complex numbers is given by the identity

$$|a + b|^2 = |a|^2 + |b|^2 + 2\text{Re}(ab^*)$$

Therefore we can write the total power as the sum of three terms:

$$P = \frac{1}{2}R_{11}|I_1|^2 + \frac{1}{2}R_{22}|I_2|^2 + R_{12}\text{Re}(I_1 I_2^*) \tag{12}$$

where

$$R_{11} = \int_{V_1 + V_2} \sigma(x, y, z)\left|\vec{E_1}(x, y, z)\right|^2 dV \tag{13}$$

$$R_{22} = \int_{V_1 + V_2} \sigma(x, y, z)\left|\vec{E_2}(x, y, z)\right|^2 dV \tag{14}$$

$$R_{12} = \int_{V_1 + V_2} \sigma(x, y, z)[\vec{E_1}(x, y, z) \cdot \vec{E_2}^*(x, y, z)]dV. \tag{15}$$

Even though $\vec{E}_1$ and $\vec{E}_2$ are complex-valued, the quantity R_{12} in (15) can be shown to be a *real* number. This permits us to drop the $\mathrm{Re}(\ )$ operator from the expression.

We now consider the meaning of the total power expression in (12). The first two terms are familiar: the coefficients R_{11} and R_{22} are the self-resistances of conductors 1 and 2, which we could have computed from formula (11) by turning on one current source (I_1 or I_2) at a time. The third term, however, can be a bit surprising: it clearly represents an additional power dissipation that occurs only if I_1 and I_2 are *both* non-zero. The physical interpretation is that the electric fields $\vec{E}_2$ produced by application of current source I_2 interact with the (direct and induced) currents $\vec{J}_1 = \sigma\vec{E}_1$ produced by the application of the source I_1 to produce some extra power loss.

The constant R_{12} has units of Ohms, and is known as the *mutual resistance* between the two conductors.

3.1. Negative Mutual Resistance

It is possible—in fact, quite common—to encounter situations where the mutual resistance is a negative number. This is very reasonable. Looking back at equation (12) for the total power dissipation, and assuming for the moment that both I_1 and I_2 are positive real numbers, we see that a negative value of R_{12} would indicate that total power dissipation with both current sources active is *lower* than it would be if we added up the power values calculated with each one turned on by itself.

Of course there is a limit to how negative the mutual resistance can be. The *total* power dissipated cannot be negative (a conductor cannot generate power.) It is possible to show that $|R_{12}| < \sqrt{R_{11}R_{22}}$ is the necessary condition for keeping the total power dissipation positive.

4. An Equivalent Circuit Model for Mutual Resistance

In this section we give an interpretation of the mutual resistance in terms of circuit elements. We propose the following circuit model for the self and mutual resistance effects in a two-conductor system:

$$\begin{aligned} V_1 &= R_{11}I_1 + R_{12}I_2 \\ V_2 &= R_{12}I_1 + R_{22}I_2 \end{aligned} \tag{16}$$

An equivalent representation using the *resistance matrix* would be

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}.$$

It is important to stress that the model proposed in (16) only attempts to represent the "resistive" component of the voltage drop across the two conductors. There is an additional "inductive" component to that voltage drop, described by the self and mutual inductances of the conductors. We will ignore the inductive part for now so that we can focus attention on modeling the resistive effects.

Here we have introduced the voltage drops V_1 and V_2 that appear across the independent current sources I_1 and I_2, respectively (see Figure 1 for sign conventions.) There are many different ways that one could attempt to define "voltage" for a high-frequency situation. The problem is complicated by the presence of magnetic fields, which means that the "voltage drop" between two points depends to some extent upon the path that we take between those points. Rather than choose a particular path between the ends of the current sources, we instead demand that the voltages in our circuit model predict the same amount of power dissipation as the detailed electromagnetic field solution. The model proposed above in (16) does this, as we now show.

Consider using circuit theory to compute the total power dissipation of the model in (16). We know that the average total power will be given by

$$P = \frac{1}{2}\mathrm{Re}(I_1^* V_1) + \frac{1}{2}\mathrm{Re}(I_2^* V_2) \tag{17}$$

That is, we expect that the power dissipated by the circuit model is going to be equal to the power being delivered by the two current sources I_1 and I_2. Using (16) in this expression to eliminate V_1 and V_2 results in

$$P = \frac{1}{2}R_{11}|I_1|^2 + \frac{1}{2}R_{22}|I_2|^2 + R_{12}\mathrm{Re}(I_1 I_2^*) \tag{18}$$

This does indeed agree exactly with the expression in (12) that we derived from electromagnetics.

To achieve agreement between the predicted power loss in the circuit model and the electromagnetic solution, we had to posit an additional voltage drop in each conductor due to a current flowing in the other one. The simplest way of achieving this effect with simple two-terminal circuit elements is to add a dependent source (a current-controlled voltage source) in series with the self resistance (Figure 2.)

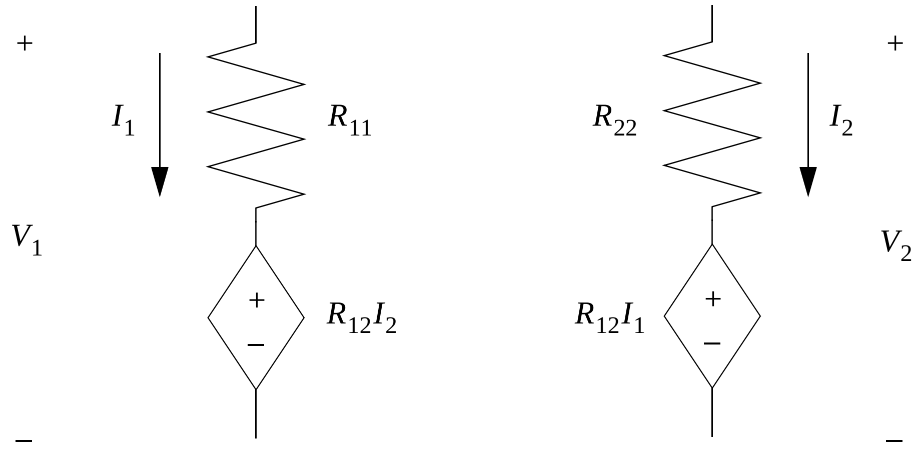

Figure 2. A straightforward circuit model of the mutual and self resistances for a two-conductor problem.

Another possible way to model the mutual resistance is to convert the circuit model in Figure 2 into a Norton equivalent model. Instead of a series-connected dependent voltage source, we end up with a parallel-connected dependent current source (Figure 3). Maxwell Spicelink uses the

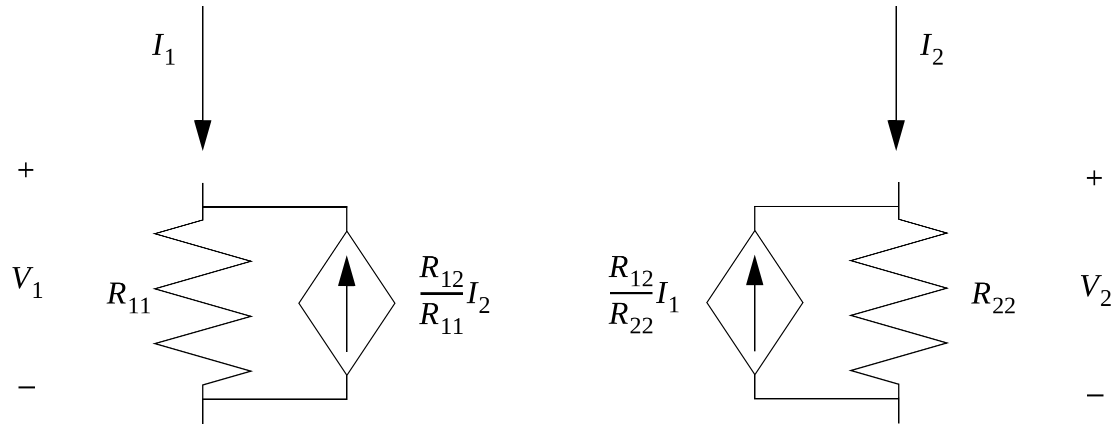

Figure 3. A Norton equivalent circuit model for the mutual resistance effect.

Norton equivalent model to produce Spice equivalent circuits with mutual resistance, because Spice can simulate these models more efficiently than the Thévenin equivalent circuits of Figure 2.

5. Common Mistakes in Using Mutual Resistance

This section discusses a few common mistakes encountered when dealing with the concept of mutual resistance.

5.1. Zero Mutual Resistance

The term "mutual resistance" sometimes gives rise to confusion because the person hearing it envisions two resistors with an additional resistor connected between them. This picture is incorrect. Once a person falls into this fallacy, he may conclude that an "infinite" value of mutual resistance is required in order to maintain electrical isolation between the two conductors.

As we have seen, the presence of a mutual resistance between two conductors does not imply the existence of a conductive path between them; rather, it indicates that the two conductors influence the power dissipation in one another through induced eddy currents. In the special case where $R_{12} = 0$, this does not mean that there is a zero-resistance connection between the two conductors. It simply indicates that no power loss or voltage drop is induced on one conductor by the other. This situation is encountered often at DC, when the resistance matrix of two separate conductors is calculated.

Indeed, we see from the circuit model of Figure 2 that the dependent voltage sources have zero gain when $R_{12} = 0$, which means that they are equivalent to short circuits in series with the self-resistances. The two conductors are still electrically isolated from one another. In the Norton equivalent model of Figure 3, the dependent sources would become open circuits, effectively removing them from the circuit. Again the two conductors remain isolated.

5.2. Mutual Resistance at DC

The mutual resistance is a frequency-dependent quantity. If we have two separate conductors, the mutual resistance between the two decreases to zero as the frequency is lowered, because the time-varying magnetic fields produced by one conductor (see equation (1)) are less and less effective at inducing eddy currents in other nearby conductors.

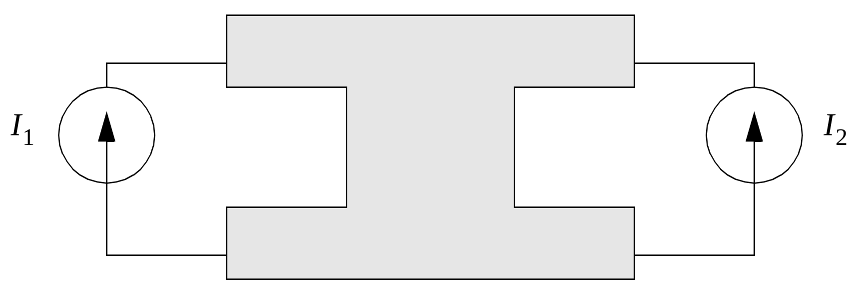

Figure 4. Mutual resistance can occur even at DC, if there are two or more conduction paths within the same conductive object.

It is possible for a non-zero mutual resistance to exist at DC—but only between two different conduction paths within a single conductor (or a collection of touching conductors.) In this case, there are of course no induced currents, and so the mutual resistance is created by interaction between direct conduction currents. Figure 4 illustrates a typical situation. As in earlier examples, the self and mutual resistances for this problem can be calculated using (13)-(15). We could choose to model this structure with the equivalent circuits from Figure 2 or Figure 3. Alterna-

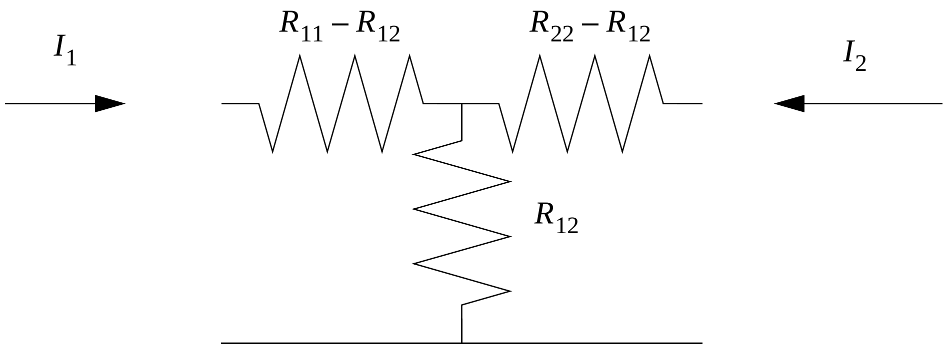

Figure 5. A T-network can be used to model the self and mutual resistance between two conduction paths within the same conductor.

tively, the T network model of Figure 5 could be used. This is permissible because there really does exist a conduction path between the two sides. However, to use the T model we must be prepared to accept negative valued resistors in our model. Also, it is difficult to generalize the T model to cases with more than 2 conductors. Therefore Maxwell Spicelink continues to use the Norton model of Figure 3 to represent such structures.

6. Where Mutual Resistance is Important

We now present an example to show why it is important to model the mutual resistance effect. The problem is a simple one: a pair of straight conductors, as shown in Figure 6 below. The conductors are copper, embedded in a vacuum. The conductors are 100 mm long, 20 mm wide and 10 mm high; the separation between them is 10 mm. This geometry was entered into Maxwell Q3D

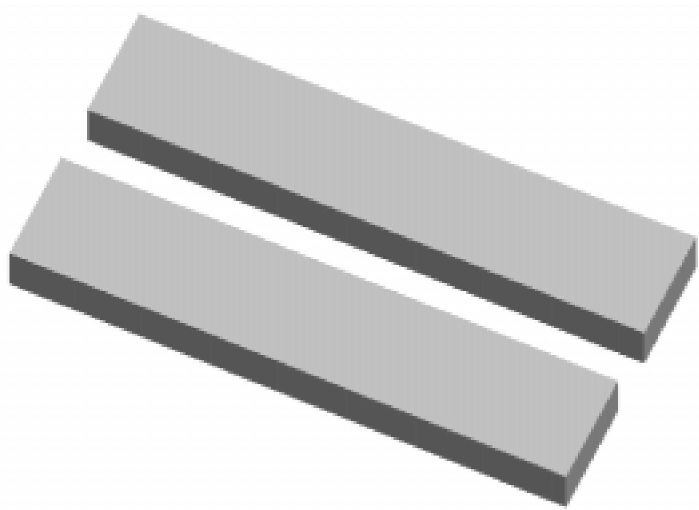

Figure 6. An example with two nearby conductors.

Extractor and solved for AC Inductance/Resistance at 100 MHz. A seeded mesh with 1224 triangles was used and 5 adaptive multipole passes were run. The resulting resistance matrix (in mΩ) was

$$R = \begin{bmatrix} 5.30 & -0.48 \\ -0.48 & 5.33 \end{bmatrix} \text{m}\Omega$$

We see that the mutual resistance is negative, and in magnitude about one tenth the value of the self resistance. To see why this could be significant, we consider the problem of calculating the resistance formed by closing the loop between the two conductors as shown schematically in Figure 7.

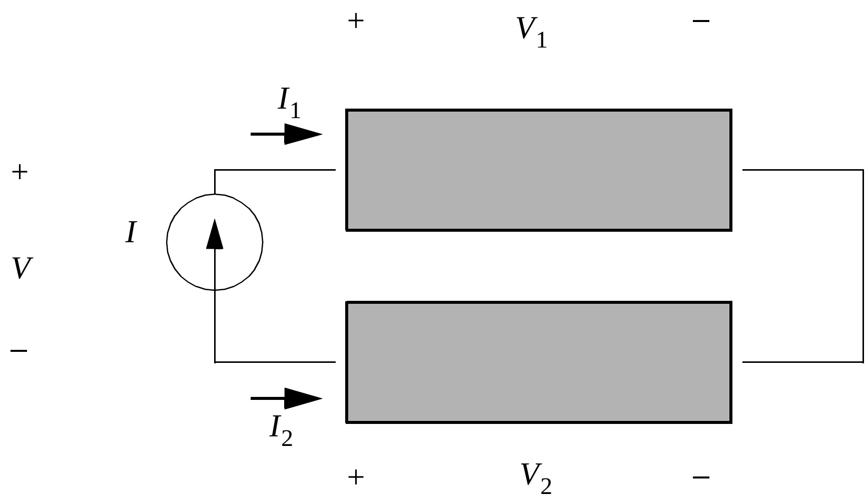

Figure 7. The two conductors are shorted at one end and driven by a current source I at the other in order to compute the overall loop resistance.

We apply the current source I and measure the resulting voltage drop V; the ratio between the two will be the resistance of the loop. Due to the connections made, we have

$$V = V_1 - V_2$$

$$I_1 = -I_2 = I$$

Therefore the voltage is

$$V = (R_{11}I - R_{12}I) - (R_{12}I - R_{22}I) = (R_{11} - 2R_{12} + R_{22})I$$

and thus the overall resistance is $R = R_{11} - 2R_{12} + R_{22}$, or 11.59 mΩ. This should be compared to the value that we would have obtained if we had ignored the mutual resistance and simply added the self resistances: $R_{11} + R_{22}$ is just 10.63 mΩ. The resulting error is on the order of 10%.

It is interesting to note that the effect of the negative mutual resistance is to *add* to the overall resistance of the loop. This should be contrasted with what occurs in calculating the loop inductance of the structure: the partial mutual inductance between the conductors is positive, which implies that it contributes to a reduction in loop inductance.

7. Summary and Conclusions

This note has discussed the mutual resistance phenomenon. We reviewed the electromagnetic theory that gives rise to the mutual resistance, and introduced circuit models that describe this effect. We also used an example to point out that the mutual resistance effect is not only of theoretical interest; correct accounting for mutual resistance can significantly improve the accuracy of total resistance calculations.

In our examples, we have focused solely on 1 and 2-conductor problems, but the results we have presented and the circuit models we have described can be generalized easily to systems with arbitrary numbers of conductors.